GENERIC ALGEBRAS WITH INVOLUTION OF DEGREE 8m

David J. Saltman*
Department of Mathematics
The University of Texas
Austin, Texas 78712

and

Jean - Pierre Tignol*
Institut de Mathématique Pure et Appliquée
Université Catholique de Louvain
Louvain-la-Neuve, Belgium

Abstract The centers of the generic central simple algebras with involution are interesting objects in the theory of central simple algebras. These fields also arise as invariant fields for linear actions of projective orthogonal or symplectic groups. In this paper, we prove that when the characteristic is not 2, these fields are retract rational, in the case the degree is $8m$ and m is odd. We achieve this by proving the equivalent lifting property for the class of central simple algebras of degree $8m$ with involution. A companion paper ([S3]) deals with the case of m, $2m$ and $4m$ where stronger rationality results are proven.

AMS Subject Classification: 16K20, 12E15, 14L24, 14L30

Key Words: Orthogonal group, Symplectic group, invariant field, rational

*The first author is grateful for support under NSF grant DMS-9970213. The first author would also like to thank the Université Catholique de Louvain for its hospitality. The second author is supported in part by the National Fund for Scientific Research (Belgium) and by the TMR network "Algebraic K - Theory, Linear Algebraic Groups and Related Structures" (ERB FMRX CT 97-0107)

In this paper F will always be an infinite field of characteristic not 2. Let $\mathcal{G}$ be an algebraic group over F and V an algebraic F representation, by which we mean there is an algebraic group morphism $\mathcal{G} \to GL_F(V)$. There is considerable interest in the structure, and more specifically in the rationality, of the invariant field $F(V)^{\mathcal{G}}$, where $\mathcal{G}$ has its natural action on the field of rational functions $F(V)$ of V. For specific groups and V, this question has particular significance. For example, consider $\mathcal{G} = PGL_n(F) = GL_n/F^*$ and $V = M_n(F) \oplus \ldots \oplus M_n(F)$ (r times) where the action of $PGL_n(F)$ on V is induced by diagonal conjugation. Then the invariant field $F(V)^{PGL_n}$ is the center of a generic division algebra $UD(F,n,r)$ (e.g. [LN] sec. 14).

In PGL_n there are subgroups and for some of these subgroups the corresponding invariant field is also of importance. We will be particularly interested in the projective orthogonal groups PO_n and projective symplectic groups PSp_n (for n even). Since we do not assume F is algebraically closed, let us be precise here. Let $O_n(F) \subset GL_n(F)$ be the group of orthogonal matrices. That is, $O_n(F)$ is the group of matrices where $AA^T = I$, where T is the transpose. Let $Sp_n(F)$ be the group of symplectic matrices, that is the group of matrices where $AA^S = I$ and S is the standard symplectic involution. For our purposes we can then define $PO_n(F)$ and $PSp_n(F)$ to be the image of $O_n(F)$ and $Sp_n(F)$ in $PGL_n(F)$. Note that, with this choice, $PO_n(F)$ and $PSp_n(F)$ may not be the group of F rational points of the corresponding algebraic group, because the quotient groups may have F points not in the image of the group of F points of O_n or Sp_n. To remedy this one could replace O_n and Sp_n by GO_n and GSp_n, the corresponding groups of similitudes (e.g. [K-T] p. 153). However, for our purposes none of this matters. Our definition of $PO_n(F)$ and $PSp_n(F)$ yield a Zariski dense set of points in the corresponding groups over the algebraic closure of F, and so the invariant rings and fields are the same no matter what definition we take.

In, for example, [R1] p. 183 there is a definition of generic algebras $UD_t(F,n,r)$ and $UD_s(F,n,r)$ with involution of orthogonal respectively symplectic type. By [P] p. 377-378, $F(V)^{PO_n}$ is the center, $Z_t(F,n,r)$, of $UD_t(F,n,r)$ while $F(V)^{PSp_n}$ is the center, $Z_s(T,n,r)$, of $UD_s(F,n,r)$. Thus the invariant fields of PO_n and PSp_n play the role in the theory of central simple algebras with involution that the invariant field of PGL_n plays in the theory of central simple algebras. In particular, these invariant fields are natural objects to consider.

Though the original question we asked was about rationality, there is a weaker property which is closely tied to properties of central simple algebras. We say a field extension K/F is **retract rational** if and only if the following holds. K is the field of fractions $q(S)$ of an F algebra domain S, and there is a localized polynomial ring $F[\vec{x}](1/s) = F[x_1, \ldots, x_n](1/s)$ with F algebra maps $f : S \to F[\vec{x}](1/s)$ and $g : F[\vec{x}](1/s) \to S$ such that $g \circ f : S \to S$ is the identity.

The basic properties of retract rational field extensions are developed in [S]. Let us note one here. Define K, K' to be stably isomorphic (over F) if and only if the following holds. For some a,b, the fields $K(x_1, \ldots, x_a)$ and $K'(y_1, \ldots y_b)$ are isomorphic over F, where the x's and y's ate transcendence bases. It is shown in [S] that if K, K' are stably isomorphic, and K/F is retract rational, then K'/F is retract rational. In particular, stably rational (i.e. stably isomorphic to a rational extension) implies retract

rational (but not conversely). Because of the above fact, we will talk about the retract rationality of the stable isomorphism class of a field extension K/F.

Let us break to explain a little notation. The statement A/K is a central simple algebra of degree n means that A is a simple algebra of dimension n^2 over its center K. If we say D/K is a division algebra, we also mean K is its center. If A/K is central simple, we will write $K(A)$ to mean the function field of the Severi Brauer variety of A. That is, $K(A)$ is the Amitsur generic splitting field of A. Finally, suppose A/K and A'/K' are central simple algebras and $K(x_1, \dots, x_a) \cong K'(y_1, \dots, y_b)$ as in the definition of stable isomorphism. If some such isomorphism extends to an isomorphism $A \otimes_K K(x_1, \dots, x_a) \cong A' \otimes_{K'} K'(y_1, \dots, y_b)$, we say A/K and A'/K' are stably isomorphic.

As mentioned above, $F(V)^{PO_n}$ and $F(V)^{PSp_n}$ are the centers of the so called generic algebras with orthogonal respectively symplectic involution. In particular, these fields are centers for generic objects for the class of central simple algebras with orthogonal respectively symplectic involutions. It follows that these are also generic objects for the class of central simple algebras of order dividing 2 in the Brauer group. This last fact is reflected in the result from [BS] we are about to quote in Theorem 1, describing $F(V)^{PO_n}$ and $F(V)^{PSp_n}$ as extensions of $F(V)^{PGL_n}$. Furthermore, in Theorem 2, we will confront more precisely what it means to be a generic object for a class of central simple algebras.

To state it the result from [BS] we need, let r be the number of direct summands in V and $UD(F,n,r)/Z(F,n,r)$ the generic division algebra of degree n in r variables. Abbreviate $UD/Z = UD(F,n,r)/Z(F,n,r)$. Let B_o be the central simple algebra of degree $n(n+1)/2$ in the Brauer class of $UD \otimes_Z UD$ and B_s the central simple algebra in the same class of degree $n(n-1)/2$. Note that B_o is written s^2UD and B_s is written $\lambda^2 UD$ in [K-T] p. 33.

Theorem 1. *For any n, $F(V)^{PO_n} = Z_t(F,n,r) = Z(B_o)$. If n is even (so PSp_n is defined), $F(V)^{PSp_n} = Z_s(F,n,r) = Z(B_s)$.*

Let D' be the division algebra in the class of $UD \otimes_Z UD$. Then by e.g. [LN] p. 93, $Z(B_o)$ and $Z(B_s)$ are, when defined, rational over $Z(D')$. In particular, $Z(B_o)$ is isomorphic to a field rational over $Z(B_s)$. Thus, to save ink, we will frequently only discuss $F(V)^{PO_n} = Z_t(F,n,r)$ since the other field is equivalent.

The goal of this note is a result on retract rationality, which we prove by relating retract rationality to a property of algebras. To this end, let $\mathcal{A}_{2,n}$ be the class of Azumaya algebras A/R of degree n where $R \supset F$ and $A \otimes_R A \cong M_t(R)$ for the appropriate t. Note that this is a linear class in the sense of [LN] p. 76. We say $\mathcal{A}_{2,n}$ has the lifting property ([LN] p. 77) if and only if the following holds. Assume T is a local commutative F algebra with residue field K and A/K is in $\mathcal{A}_{2,n}$. Then there is an Azumaya $B/T \in \mathcal{A}_{2,n}$ with $B \otimes_T K \cong A$.

Lifting is important because of Theorem 2 to follow. But before we state the result, we recall a few notions from [LN] section 11. $UD_t = UD_t(F,n,r)$ can be identified with $UD \otimes_Z Z(B_o)$ and the center of both these algebras can be identified with $F(V)^{PO_n}$. Suppose A/S is an Azumaya such that $q(S) = F(V)^{PO_n}$ and $A \otimes_S F(V)^{PO_n} = UD_t$. If $B/R \in \mathcal{A}_{2,n}$, we say $\phi : S \to R$ realizes B if and only if $B \cong A \otimes_\phi R$. Note that $\otimes_\phi$ means that we treat R as an S module via ϕ.

We say $UD_t/Z(B_o)$ represents $\mathcal{A}_{2,n}$ (see [LN] p. 76) if and only if the following holds. There is an A/S Azumaya such that S is finitely generated as an F algebra, $q(S) = Z(B_o)$, $A \otimes_S Z(B_o) \cong UD_t$, and further the following holds. Assume $0 \neq s \in S$ and $B/K \in \mathcal{A}_{2,n}$ with K a field. Then there is a $\phi : S(1/s) \to K$ realizing B/K. Note that if A/S is as above, and $S' \subset F(V)^{PO_n}$ satisfies $q(S') = F(V)^{PO_n}$, then for some $0 \neq s' \in S'$ and some $A'/S'(1/s')$, $A'/S'(1/s')$ satisfies the same property. This is why we can view "representing" as a property of the algebra $UD_t/Z(B_o) = UD_t/F(V)^{PO_n}$. Also, it is clear that if $UD_t/F(V)^{PO_n}$ is stably isomorphic to a A/K, and A/K represents $\mathcal{A}_{2,n}$, then so does $UD_t/F(V)^{PO_n}$. Thus we can talk of the stable isomorphism class of $UD_t/F(V)^{PO_n}$ as representing $\mathcal{A}_{2,n}$.

Another idea we recall is called "local projectivity" in [S], or (a slight variant) property v) in [LN] p. 76. We will use the version of this property from [LN], but the name local projectivity from [S]. Let A/S be such that $q(S) = F(V)^{PO_n}$ and $A \otimes_S F(V)^{PO_n} = UD_t$. Suppose $B'/T \in \mathcal{A}_{2,n}$ and T is a local ring with residue field K. Set $B = B' \otimes_T K$. Then A/S is locally projective if and only if for any such B'/T etc., and any $\phi \to K$ realizing B/K, there is a $\phi' : S \to T$ realizing B'/T such that the composition $S \to T \to K$ is ϕ. Note that if A/S is locally projective then so is $A(1/s)/S(1/s)$ for any $0 \neq s \in S$. Thus once again it is fair to talk about $UD_t/F(V)^{PO_n}$ being locally projective. Also it is clear that the property of being locally projective is preserved by stable isomorphisms. Thus, once again, we can talk about the stable isomorphism class of $UD_t/F(V)^{PO_n}$ as being locally projective.

In [S] and [LN sec. 11] a general framework is described along with a result connecting lifting properties with retract rationality. This framework applies here and so we can show:

Theorem 2. *The stable isomorphism classes of $F(V)^{PO_n}/F = Z_t(F, n, r)/F$ or (when n even) $F(V)^{PSp_n}/F = Z_s(F, n, r)/F$ are retract rational if and only if $\mathcal{A}_{2,n}$ has the lifting property.*

Proof. By [LN] p. 77 it is enough show that $UD_t/F(V)^{PO_n}$ represents $\mathcal{A}_{2,n}$ and is locally projective. By the above observations, we can replace $Z(B_o) = F(V)^{PO_n}$ by $K = Z(UD \otimes_Z UD)$, and UD_t by $D = UD_t \otimes_{Z(B_o)} K$, because $K/Z(B_o)$ is rational (e.g. [LN] p. 93).

In [S1] was defined a generic central simple algebra D'/K' of degree n and order dividing t. In that paper D'/K' was shown to represent the class of Azumaya algebras with the same property. In the case of $t = 2$, it follows from [S2] p. 344 that D'/K' is rational over D/K, and so D/K represents $\mathcal{A}_{2,n}$.

In [LN] p. 105 it is shown that UD/Z is locally projective for the class of Azumaya algebras of degree n. Let A'/S', $q(S') = Z$, be an Azumaya algebra that realizes this property. Define $S \supset S'$ to be the affine ring of an affine open subset of the Severi-Brauer scheme of $A' \otimes_{S'} A'$ (e.g. [V]) and set $A = A' \otimes_{S'} S$. Then $q(S) = Z(UD \otimes_Z UD) = K$ by the naturality of the Severi-Brauer scheme. Furthermore, clearly $A \otimes_S K = D$. We claim that using A/S one sees that D/K is locally projective.

Suppose B'/T is in $\mathcal{A}_{2,n}$, T is local with residue field K, and $B = B' \otimes_T K$. Assume $\phi : S \to K$ realizes B/K. Since A'/S' is locally projective, there is a partial lifting $\phi'' : S' \to T$ which realizes B'. That is, the restriction $\phi|_{S'} : S' \to K$ can be

factored into $S' \to T \to K$ where the first map is ϕ''. The full map ϕ can be factored into $S \to S \otimes_{\phi''} T \to K$. Note that by the naturality of the Severi Brauer scheme, $S \otimes_{\phi''} T$ is the affine ring of the corresponding open subset, call it U, of the Severi Brauer scheme of $B' \otimes_T B'$. Thus ϕ defines a K point on the Severi-Brauer variety of $B' \otimes_T B'$ which can be identified with a K point of the Severi Brauer variety of $B \otimes_K B$. There is a transitive action by $(B \otimes_K B)^*$ on these K points, and $(B' \otimes_T B')^*$ maps onto $(B \otimes_K B)^*$. By assumption, there is a T point on the Severi-Brauer scheme of $B' \otimes_T B'$. It follows that the K point given by ϕ is the image of a T point of the Severi-Brauer scheme of $B' \otimes_T B'$. Since T is local, the closure of this T point includes the ϕ given K point, and so this T point is also in U. That is, there is a morphism $S \otimes_{\phi''} T \to T$ and the composition $\phi' : S \to S \otimes_{\phi''} T \to T$ is the required lift for ϕ. This proves local projectivity and hence Theorem 2.

It is clear how we will use Theorem 2, but before we do that let us make one final reduction.

Lemma 3. *Let $n = 2^r m$ where m is odd. Then $\mathcal{A}_{2,n}$ has the lifting property if $\mathcal{A}_{2,2^r}$ has the lifting property.*

Proof. If A/K is in $\mathcal{A}_{2,n}$, then $A = A_2 \otimes A_m$ where A_2 has degree 2^r and A_m has degree m (e.g. [LN] p. 35). Since A has order 2 in the Brauer group, and A_m has order dividing m, it follows that A_m must be split. That is, $A \cong M_m(A_2)$. It is now obvious that if $\mathcal{A}_{2,2^r}$ has the lifting property then so does $\mathcal{A}_{2,n}$. ∎

We remark that the converse is also true, but to prove this would take us too far afield. To outline the argument, if B/T is an Azumaya algebra over a local ring, then $B \cong M_s(D)$ where D has no nontrivial idempotents. Moreover, there is only one such D, up to isomorphism, in the Brauer class of B. With this, one can copy the usual proof over a field, and show that $B \cong B_1 \otimes_T \ldots \otimes_T B_s$ where all the B_i have prime power degree. With this background, the converse is clear.

We can now state:

Theorem 4. *Suppose F is an infinite field of characteristic not 2 and $n = 8m$ where m is odd. Then the stable isomorphism classes of $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are retract rational over F. Equivalently, the stable isomorphism classes of the centers $Z_t(F,n,r)$ and $Z_s(F,n,r)$ of the generic algebras with orthogonal respectively symplectic involution are retract rational over F.*

Before we prove Theorem 4, we begin with another lemma. Let R be a commutative ring. If $b_i \in R$ are finitely many elements, define $R(b_1^{1/2}, \ldots, b_s^{1/2})$ to be $R[x_1, \ldots, x_s]/ < x_i^2 - b_i | i = 1, \ldots s >$. Note that we make the above definition even if some of the b_i are squares. In particular, if R is a field, $R(a_1^{1/2}, \ldots, a_s^{1/2})$ may not be a field but is a direct sum of fields. We recall:

Lemma 5. *Let T be a local F algebra with residue field K. Suppose $a_i \in K^*$ and $a'_i \in T$ are preimages. Then $S = T(a_1'^{1/2}, \ldots, a_s'^{1/2})$ is a semilocal F algebra which, modulo its Jacobson radical, is isomorphic to $L = K(a_1^{1/2}, \ldots, a_s^{1/2})$. In particular, S^**

maps onto L^*. S/T is Galois with Galois group we can identify with the Galois group of L/K. Call this group G. There is an isomorphism $H^2(G, S^*) \cong \mathrm{Br}(S/T)$.

Proof. Since the a_i' are invertible, it is easy to see S/T is Galois and since Galois extensions are closed under specialization, one can identify this Galois group with that of L/K. The Jacobson radical of S must be $\mathcal{M}S$ where $\mathcal{M}$ is the maximal ideal of T. Since L is a direct sum of fields, S is semilocal. Of course, semilocal local rings have trivial Picard group, so $H^2(G, S^*) \cong \mathrm{Br}(S/T)$ by, e.g., [LN] p.45. ∎

If A' is any T algebra, and T has residue field K, then we say A' is a lift of $A = A' \otimes_T K$. When A/K is central simple, we will only call A' a lift if A'/T is Azumaya. When A/K is a commutative Galois extension with Galois group G, we will only say A' is a lift if A'/T is Galois with group G. Thus among the results of Lemma 5 is that $T(a_1'^{1/2}, \ldots, a_s'^{1/2})$ is a lift of $K(a_1^{1/2}, \ldots, a_s^{1/2})$.

Let us also recall that if R is any commutative ring containing $1/2$, and $a, b \in R^*$, then one can form the Azumaya quaternion algebra $(a,b)_R = R \oplus R\alpha \oplus R\beta \oplus R\alpha\beta$ where $\alpha^2 = a$, $\beta^2 = b$, and $\alpha\beta = -\beta\alpha$. As implied, $(a,b)_R$ is Azumaya over R of rank 4 (i.e. degree 2) ([LN] p. 49). By e.g. [LN] p. 34, (a,b) defines an element of order 2 in the Brauer group of R. Furthermore, $(a,b)_R \cong (b,a)_R \cong (a, N_S(\gamma)b)_R$ where $\gamma \in R(a^{1/2})^*$, $S = R(a^{1/2})$, and $N_S : R(a^{1/2}) \to R$ is the norm. If R is semilocal, then $(a,b) \cong (a,c)$ implies bc is a norm from $R(a^{1/2})$ by Lemma 5.

Let $a \in R^*$ with R as above, and $S = R(a^{1/2})$. Then the corestriction $\mathrm{Cor}_{S/R} : \mathrm{Br}(S) \to \mathrm{Br}(R)$ is defined (e.g. [LN] p. 55) and satisfies all the usual properties. In particular, if $a \in R^*$ and $b \in S^*$, then $\mathrm{Cor}_{S/R}((a,b)_S)$ is Brauer equivalent to $(a, N_S(b))_R$ (e.g. [LN] p. 57). Furthermore, if A/R is Azumaya, $\mathrm{Cor}_{S/R}(A \otimes_R S)$ is Brauer equivalent to $A \otimes_R A$. Let σ generate the Galois group of S/R. That is, $\sigma(a^{1/2}) = -a^{1/2}$. Suppose B/S is Azumaya and let $\sigma(B)$ be the σ twist. That is, $\sigma(B) = B \otimes_\sigma S$. We finally have $\mathrm{Cor}_{S/R}(B) \otimes_R S$ is Brauer equivalent to $B \otimes_S \sigma(B)$.

We are finally ready to turn to the proof of Theorem 4. Of course, by Theorem 2 and Lemma 3 it suffices to prove $\mathcal{A}_{2,8}$ has the lifting property. To this end, suppose T is a local F algebra with residue field K, and D/K is a central simple algebra of degree 8 and order 2 in the Brauer group. We must show that there is an Azumaya D'/T such that $D' \otimes_T K \cong D$ and $D' \otimes_T D'$ is isomorphic to matrices over T. Note that since T is local, this is equivalent to saying D' has order dividing 2 in the Brauer group.

By [R], D has a maximal subfield of the form $K(a_1^{1/2}, a_2^{1/2}, a_3^{1/2})$. The centralizer of $L = K(a_1^{1/2})$ in D is a division algebra of degree 4 with involution. Thus by e.g. [LLT] Proposition 5.2, this centralizer has the form $B = (a_2, x_2)_L \otimes_L (a_3, x_3)_L$.

The corestriction of $[B]$ is Brauer equivalent to $D \otimes_K D$ and so must be trivial. But this corestriction is $(a_2, N_L(x_2))_K \otimes_K (a_3, N_L(x_3))_K$. In other words, $(a_2, N_L(x_2))_K \cong (a_3, N_L(x_3))$. By [T] p.267 or [A] Lemma 1.7, there is a $y \in K^*$ such that $(a_2, N_L(x_2)) \cong (y, N_L(x_2)) \cong (y, N_L(x_3)) \cong (a_3, N_L(x_3))$. Set $L_i = K(N_L(x_i)^{1/2})$ for $i = 2, 3$ and $L_{23} = K(N_L(x_2 x_3)^{1/2})$. Then there are $\mu_i \in L_i^*$ and $\mu_{23} \in L_{23}^*$ such that $a_2 y = N_{L_2}(\mu_2)$, $y = N_{L_{23}}(\mu_{23})$, and $a_3 y = N_{L_3}(\mu_3)$. The idea of this proof is that we can lift a_1, then the x_i, then y, and then a_2, a_3 so that all these relations still hold. The key idea is that we use the relations to define the lifts.

Choose $a_1' \in T^*$ a preimage of a_1. Set $S = T(a_1'^{1/2})$, so S is a lift of L. Choose $x_i' \in S^*$ preimages of the x_i. Of course, $N_S(x_i')$ is a preimage of $N_L(x_i)$. Set $S_i = T(N_S(x_i')^{1/2})$ and $S_{23} = T(N_S(x_2'x_3')^{1/2})$. Of course, the S_i and S_{23} are lifts of the L_i and L_{23} respectively. Choose $\mu_i' \in S_i^*$ and $\mu_{23}' \in S_{23}^*$ preimages of the μ_i and μ_{23}' respectively.

Set $y' = N_{S_{23}}(\mu_{23}')$. Clearly $y' \in T^*$ is a preimage of y. For $i = 2, 3$, set $a_i' = N_{S_i}(\mu_i')y'^{-1} \in T^*$. Clearly, the a_i' are preimages of the a_i. Set $B' = (a_2', x_2')_S \otimes_S (a_3', x_3')_S$. Of course, B' is a lift of B. The corestriction $\mathrm{Cor}_{S/T}(B')$ is Brauer equivalent to $(a_2', N_S(x_2'))_T \otimes_T (a_3', N_S(x_3'))_T$. But $(a_2', N_S(x_2')) \cong (y', N_S(x_2')) \cong (y', N_S(x_3')) \cong (a_3', N_S(x_3'))$. It follows that $\mathrm{Cor}_{S/T}(B')$ is trivial. Tensoring up to S, we have $B' \otimes_S \sigma(B')$ is trivial where σ generates the Galois group of S/T. Of course this means B' and $\sigma(B')$ are Brauer equivalent. Since S is semilocal, using [D] we have that $B' \cong \sigma(B')$. Alternatively, we can make the following argument. Both B' and $\sigma(B')$ are split by $V = S(a_2'^{1/2}, a_3'^{1/2})$. More precisely, both B' and $\sigma(B')$ are crossed products (e.g. [OS] p. 88-90) with respect to V/S. By [LN] p. 45, the corresponding cocycles are cohomologous, and so $B' \cong \sigma(B')$.

The isomorphism $B' \cong \sigma(B')$ can be equivalently expressed as the existence of an $\alpha : B' \cong B'$ such that α is σ semilinear. Since α^2 is an S automorphism, and S is semilocal, α^2 is an inner automorphism given by, say, $c \in B'^*$ (e.g. [LN] p. 16).

Form the algebra $A' = B' \oplus B'u$ where $ub = \alpha(b)u$ for all $b \in B'$ and $u^2 = c$. Using e.g. [LN] p. 12 it is easy to see that A'/T is Azumaya over T of degree 8, and the centralizer, in A', of $S \subset B'$ is B'. Thus (e.g. [LN] p. 24) A'/T defines a preimage of B' in the Brauer group of T. In particular, $A' \otimes_T A'$ is Brauer equivalent to $\mathrm{Cor}_{S/T}(B')$ and so A' has order 2 in the Brauer group.

If $A = A' \otimes_T K$, then A and D have equal images in the Brauer group of L. That is, $M_2(A) \cong D \otimes_K (a_1, d)$ for some $d \in K^*$. Let $d' \in T$ be a preimage of d and set $A'' = A' \otimes_T (a_1', d')$. Of course, the Brauer class of A'' is a preimage of the Brauer class of D. A'' contains the subalgebra $S \otimes_T S$. Since S/T is Galois, $S \otimes_T S$ contains an idempotent e such that $e(S \otimes_T S) \cong S$. Viewing $e \in A''$, it is easy to see that $D' = eA''e$ is Azumaya over T of degree 8 and so D' is a lift of D. ■

References

[A] Arason, J.K., *Cohomologische Invarianten Quadratischen Formen*, J. Algebra **36** (1975), 448–491.

[BS] Berele, A., and Saltman, D.J., *The centers of generic division algebras with involution*, Israel J. Math. (1) **63** (1988).

[D] DeMeyer, F.R., *Projective modules over central separable algebras*, Canad. J. Math. **21** (1969), 39–43.

[K-T] Knus, M.A., Merkurjev, A., Rost, M., and Tignol, J.-P., *The Book of Involutions*, Amer. Math. Soc., Providence, RI, 1998.

[LLT] Lam, T,Y., Leep, D.B., and Tignol, J.-P., *Biquaternion algebras and quartic extensions*, Pub. Math. IHES **77** (1993), 63–102.

[LN] Saltman, D.J., *Lectures on Division Algebras*, Amer. Math. Soc., Providence, RI, 1999.

[P] Procesi, C., *The invariant theory of $n \times n$ matrices*, Advances in Math. **19** (1976), no. 3, 306–381.

[OS] Orzech, S., and Small, C., *The Brauer group of commutative rings*, Marcel Dekker, New York, 1975.

[R] Rowen, L.H., *Central simple algebras*, Israel J. Math. **29** (1978), 285–301.

[R1] Rowen, L.H., *Polynomial identities in ring theory*, Academic Press, New York, London, 1980.

[S] Saltman, D.J., *Retract rational fields and cyclic Galois extensions*, Israel J. Math. (2-3) **47** (1984), 165–215.

[S1] Saltman, D.J., *Indecomposable division algebras*, Comm. Algebra, (8) **7** (1979), 791–817.

[S2] Saltman, D.J., *Norm polynomials and algebras*, J. of Algebra, **62** (1978), 333–345.

[S3] Saltman, D.J., *Invariant fields of symplectic and orthogonal groups*, preprint

[T] Tate, J., *Relations between K_2 and Galois Cohomology*, Invent. Math. **36** (1976) 257–274.

[V] Van den Bergh, M., *The Brauer-Severi scheme of the trace ring of generic matrices*, Perspectives in ring theory (Antwerp 1987), 333–338; Nato Adv. Sci. Inst. Ser. C: Math. Phys.Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988.